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Impact-induced phase transitions in thermoelastic solids

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Contents

A continuum model of the macroscopic behaviour of solids capable of undergoing displacive phase transitions is applied to determine the response of such materials to mechanical loading by impact. The solid is modelled using one-dimensional finite thermoelasticity, and the model incorporates both a kinetic relation and a nucleation criterion controlling the evolution and initiation of the phase transition, respectively.

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1. Introduction

We make use of recently developed continuum models of phase transitions in solids (Abeyaratne & Knowles 1993, 1994*a*, *b*) to study the dynamics of impact-generated response in materials capable of undergoing such transitions. Such materials include shape-memory alloys (CuAlNi, for example) and many ceramics (GeO_2, SiO_2). There is a substantial literature on experiments in which a flyer plate or projectile strikes a specimen and induces a phase change (see, for example, Jackson & Ahrens 1979; Tan & Ahrens 1990; Escobar & Clifton 1993, 1995; Clifton 1993; Grady 1995). The analysis presented here concerns a one-dimensional continuum composed of a thermoelastic

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material whose bulk behaviour is governed by an explicit two-well Helmholtz free energy potential that describes a material capable of existing in two distinct solid phases. This potential is an adaptation to compression of one that was introduced in Abeyaratne & Knowles (1993) for the purpose of modelling the behaviour of specimens in which phase changes occur in tension. The thermomechanical setting in the present study assumes that the relevant dynamical process takes place adiabatically, as in the theory put forward in Abeyaratne & Knowles (1994). Moreover, the model incorporates a kinetic relation controlling the rate at which the phase transition proceeds as well as a nucleation criterion that determines when such a transition is initiated. A different analysis, designed to model experimentally observed shock-induced phase changes in materials of geological interest, is presented by Swegle (1989). Studies of the dynamics of solid–solid phase transitions in a purely mechanical framework that are similar in spirit to the present work, but without thermal effects, have been carried out by Lin (1995) and Zhong (1995).

In the next section, we give the basic thermomechanical field equations and jump conditions for a one-dimensional continuum, without yet specifying any constitutive law. In § 3, we describe the bulk constitutive law for one-dimensional nonlinear thermoelasticity theory in general, and we specify the particular two-phase 'trilinear' thermoelastic material, or 'two-well potential', that we shall employ. We also introduce in § 3 the kinetic relation and the nucleation criterion that comprise a part of the model. The field equations and jump conditions are specialized to the trilinear thermoelastic material in § 4. Section 5 is devoted to the formulation of the impact problem for a semi-infinite bar. In this problem, the bar is initially at a given temperature and at rest in its reference state. At time $t = 0$, it is suddenly subjected at its end to a prescribed constant particle velocity, which is maintained for all subsequent time. We also describe in § 5 the general structure to be expected of possible solutions to the impact problem. Explicit solutions corresponding to the cases in which the bar either changes phase under impact or *fails* to change phase are constructed in § 6 for a special case of the trilinear material in which the coefficent of thermal expansion α common to both phases of the material vanishes. The discussion in this section illustrates the way in which the nucleation criterion selects between these two types of solutions, and it also describes precisely how the kinetic relation picks out the proper solution from among a one-parameter family of solutions that correspond to a phase change. Section 6 also includes a comparison of the particle-velocity and density time-histories for the solutions with and without a phase change, and a discussion of the final temperature reached in the bar when the impact causes a phase transition. In § 7, we describe the modifications to the solutions of § 6 that are necessary when α does not vanish by describing the nature of the fields when a dimensionless version of α – sometimes called the *Grüneisen parameter* – is small. These modifications are mainly due to the presence of fans in the solutions, in addition to piecewise constant states; the latter are are the only types of fields possible when $\alpha = 0$.

2. Field equations and jump conditions

Consider a semi-infinite solid bar occupying the right half of the x-axis in the reference configuration. We shall study one-dimensional longitudinal motions of the bar in which a particle with Lagrangian coordinate x in the reference configuration is carried at time t to the point $x + u(x, t)$, where u is the displacement. The strain

$\gamma(x,t)$ and the particle velocity $v(x,t)$ at particle x at time t are defined by $\gamma = u_x$, $v = u_t$, where the subscripts indicate partial derivatives. To assure that the mapping $x \to x + u(x,t)$ is one-to-one, so that neither interpenetration of material nor splitting of a particle can occur, one must require that

$$\gamma(x,t) > -1 \tag{2.1}$$

for all particles x and all times t during the motion. The nominal stress acting at time t at the particle whose referential location is x is denoted by $\sigma(x,t)$. The mass density *in the reference state* is ρ_0, which is taken to be independent of x. The absolute temperature at time t of the particle x is denoted by $\theta(x,t)$, the Helmholtz free energy per unit mass by $\psi(x,t)$, and the entropy per unit mass by $\eta(x,t)$.

It is assumed throughout that the displacement $u(x,t)$ is a continuous function of x and t, but the strain $\gamma(x,t)$ and particle velocity $v(x,t)$ are permitted to jump across curves in the x,t-plane that correspond to moving strain discontinuities; as we shall see, such discontinuities may be either *shock waves* or *phase boundaries*. Moreover, because our thermodynamic framework will be adiabatic, we permit the temperature $\theta(x,t)$ to jump across such curves as well. (If heat conduction were accounted for as in Abeyaratne & Knowles (1994*a*), the temperature would be required to be continuous.) At points of the bar away from discontinuities, the fields are assumed to be smooth and to satisfy the differential equations,

$$\sigma_x = \rho_0 v_t, \tag{2.2}$$

$$v_x = \gamma_t, \tag{2.3}$$

$$\psi_t - \frac{\sigma}{\rho_0}\gamma_t + \eta\theta_t + \theta\eta_t = 0. \tag{2.4}$$

Equation (2.2) comes from balance of momentum, (2.3) from smoothness and continuity of displacement, and (2.4) represents the adiabatic version of energy balance, i.e. the first law of thermodynamics, in terms of the Helmholtz free energy density. The system (2.2)–(2.4) comprises the *Lagrangian* version of the field equations.

Where η is smooth, the second law of thermodynamics requires that

$$\eta_t \geqslant 0. \tag{2.5}$$

At a moving strain discontinuity whose position in the reference configuration is $x = s(t)$ at time t, the balance principles and smoothness provide the following (Lagrangian) jump conditions:

$$[[\sigma]] + \rho_0 \dot{s}[[v]] = 0, \tag{2.6}$$

$$[[v]] + \dot{s}[[\gamma]] = 0, \tag{2.7}$$

$$\{[[\psi]] + \langle\theta\rangle[[\eta]] + \langle\eta\rangle[[\theta]] - \langle\sigma/\rho_0\rangle[[\gamma]]\}\dot{s} = 0; \tag{2.8}$$

here we have written $[[g]] \equiv g(s(t)+,t) - g(s(t)-,t)$ and $\langle g\rangle \equiv \frac{1}{2}\{g(s(t)+,t) + g(s(t)-,t)\}$ for the jump and the average, respectively, of any field quantity $g(x,t)$ across a moving discontinuity.

At a jump, the second law requires that the entropy per unit mass of a particle cannot decrease as the particle crosses the discontinuity; thus

$$[[\eta]]\dot{s} \leqslant 0. \tag{2.9}$$

For our purposes, it is important to introduce the *driving traction* f *acting on the* *strain discontinuity*:

$$f = \rho_0\{[[\psi]] - \langle\sigma/\rho_0\rangle[[\gamma]] + \langle\eta\rangle[[\theta]]\}; \tag{2.10}$$

f is an ingredient in the kinetic relation to be discussed below. In terms of f, the energy jump condition (2.8) can be rewritten as

$$\{f + \rho_0\langle\theta\rangle[[\eta]]\}\dot{s} = 0. \tag{2.11}$$

From (2.11), one has $[[\eta]] = -f/(\rho_0\langle\theta\rangle)$ when $\dot{s} \neq 0$, from which it follows that the entropy inequality (2.9) is equivalent to

$$f\dot{s} \geqslant 0. \tag{2.12}$$

This representation of the second law at a moving strain discontinuity, though less convenient for calculation in the adiabatic case than (2.9), is of precisely the same form as that arising in the purely mechanical theory of the dynamics of phase transitions (Abeyaratne & Knowles 1991), as well as in the dynamics of thermoelastic phase transitions with heat conduction (Abeyaratne & Knowles 1994*a*). The same inequality plays a major role in the application of models of the kind considered here to the description of *quasi-static* hysteresis in shape-memory materials (Abeyaratne & Knowles 1993; Abeyaratne *et al.* 1994). The physical significance of f in the present adiabatic setting has been discussed in § 6 of Abeyaratne & Knowles (1994*b*); for further background concerning the notion of driving traction, see Abeyaratne & Knowles (1990).

3. A thermoelastic material

The bulk response of a thermoelastic material in the present one-dimensional context is characterized by a Helmholtz free energy potential $\hat{\psi}(\gamma, \theta)$, measured per unit mass, such that

$$\psi(x,t) = \hat{\psi}(\gamma(x,t), \theta(x,t)). \tag{3.1}$$

The nominal stress and the specific entropy are then determined by γ and θ according to the thermoelastic constitutive law

$$\sigma = \hat{\sigma}(\gamma,\theta) \equiv \rho_0\hat{\psi}_\gamma(\gamma,\theta), \qquad \eta = \hat{\eta}(\gamma,\theta) \equiv -\hat{\psi}_\theta(\gamma,\theta), \tag{3.2}$$

where the subscripts indicate partial derivatives. For such a material, the *isothermal elastic modulus* $\mu(\gamma,\theta)$, the *coefficient of thermal expansion* $\alpha(\gamma,\theta)$ and the *specific heat at constant strain* $c(\gamma,\theta)$ are defined by

$$\left.\begin{aligned} \mu(\gamma,\theta) &= \hat{\sigma}_\gamma(\gamma,\theta) = \rho_0\hat{\psi}_{\gamma\gamma}(\gamma,\theta), \\ \alpha(\gamma,\theta) &= -\hat{\sigma}_\theta(\gamma,\theta)/\hat{\sigma}_\gamma(\gamma,\theta) = -\hat{\psi}_{\gamma\theta}(\gamma,\theta)/\hat{\psi}_{\gamma\gamma}(\gamma,\theta), \\ c(\gamma,\theta) &= \theta\hat{\eta}_\theta(\gamma,\theta) = -\theta\hat{\psi}_{\theta\theta}(\gamma,\theta). \end{aligned}\right\} \tag{3.3}$$

For a thermoelastic material, the driving traction f of (2.10) can now be expressed in terms of the potential $\hat{\psi}$ through the following striking formula:

$$f = \rho_0\{[[\hat{\psi}]] - \langle\hat{\psi}_\gamma\rangle[[\gamma]] - \langle\hat{\psi}_\theta\rangle[[\theta]]\}. \tag{3.4}$$

We now turn to the particular thermoelastic material to be used here. To describe the Helmholtz free energy potential $\hat{\psi}$ for this material, we first divide the γ,θ-plane of figure 1 into four disjoint regions P, P_1, P_2, P_3. The region P is the quarter-plane

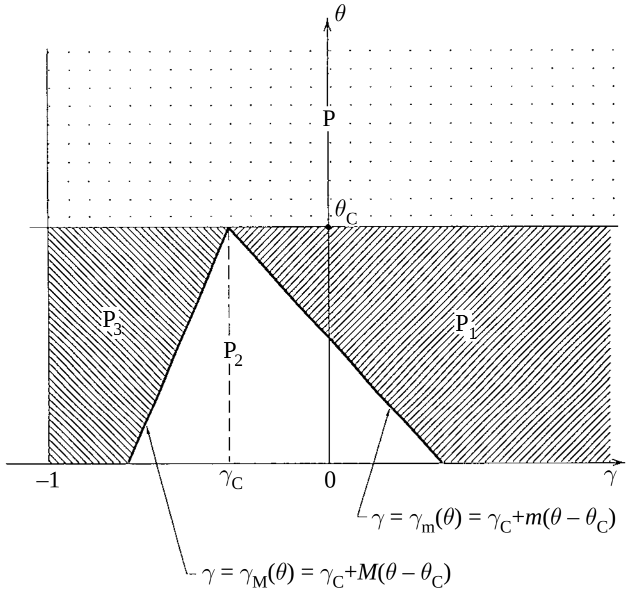

Figure 1. The strain–temperature plane.

$\gamma > -1$, $\theta \geqslant \theta_C$; the straight boundaries between P_1 and P_2 and between P_2 and P_3 are respectively given by

$$\gamma = \gamma_m(\theta) \equiv \gamma_C + m(\theta - \theta_C), \qquad \gamma = \gamma_M(\theta) \equiv \gamma_C + M(\theta - \theta_C), \tag{3.5}$$

where γ_C, $\theta_C > 0$, M and m are material constants. Since $\gamma_m(\theta) > \gamma_M(\theta)$ for $0 < \theta < \theta_C$, M and m must satisfy

$$M > m. \tag{3.6}$$

Each of the four regions P, P_1, P_2, P_3 is identified with a *phase* of the material. Above the critical temperature θ_C, the material is in the same phase regardless of the value of the strain γ. Below $\theta = \theta_C$, the material is said to be in the *low-strain phase* – the phase with lesser compression – if (γ, θ) is in P_1, in the *high-strain phase* if (γ, θ) is in P_3, and in the *intermediate phase* on P_2. On the regions P_1 and P_3, the Helmholtz free energy is given by

$$\hat{\psi}(\gamma,\theta) = \begin{cases} \dfrac{\mu}{2\rho_0}\gamma^2 - \dfrac{\alpha\mu}{\rho_0}\gamma(\theta - \theta_T) - c\theta \log \dfrac{\theta}{\theta_T} & \text{on } P_1, \\ \dfrac{\mu}{2\rho_0}(\gamma + \gamma_T)^2 - \dfrac{\alpha\mu}{\rho_0}(\gamma + \gamma_T)(\theta - \theta_T) - c\theta \log \dfrac{\theta}{\theta_T} + \lambda_T \dfrac{\theta - \theta_T}{\theta_T} & \text{on } P_3; \end{cases} \tag{3.7}$$

here $\mu > 0$, $\alpha \geqslant 0$ and $c > 0$ are constants. They represent respectively the values of the isothermal elastic modulus, the coefficient of thermal expansion and the specific heat at constant strain *common to both the low-strain and high-strain phases* of the material. (Attributing *different* values of these parameters to the two phases presents no difficulty of principle, but greatly complicates the detailed formulas.) The remaining constants $\theta_T > 0$, $\gamma_T > 0$ and λ_T are intimately connected with the physics of the low-strain-to-high-strain phase transition, or the reverse transition; their precise physical meanings will be described shortly.

Since we shall not be concerned with temperatures above θ_C, we do not discuss

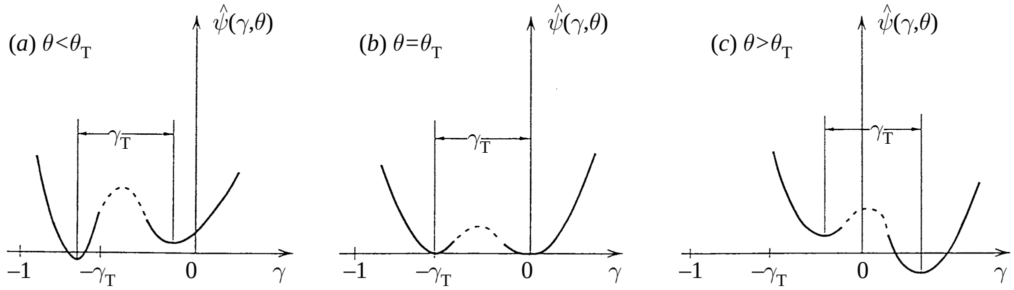

Figure 2. Helmholtz free energy potential as a function of strain at various temperatures near θ_T.

the representation of $\hat{\psi}$ on the half-plane P. It should be noted, however, that $\nabla\psi$ is necessarily discontinuous at the point (γ_C, θ_C); it is expected that the present model would not give physically meaningful predictions near this critical point, and we accordingly assume in all that follows that the temperature remains well away from θ_C.

It is also unnecessary for our purposes to display the explicit formula for $\hat{\psi}$ on the intermediate region P_2; we note only that (i) on P_2, $\hat{\psi}(\gamma,\theta)$ is quadratic in γ for each θ, with $\hat{\psi}_{\gamma\gamma} < 0$, and (ii) $\hat{\psi}$ and both of its first derivatives are continuous on $P_1 + P_2 + P_3$. The explicit formula for $\hat{\psi}$ on P_2 can be found by adapting its counterpart in Abeyaratne & Knowles (1993) to the present circumstances.

At each temperature below θ_C, this Helmholtz free energy potential is a piecewise quadratic function of strain that is convex on $(-1, \gamma_M(\theta))$ and on $(\gamma_m(\theta), \infty)$, but concave on $(\gamma_M(\theta), \gamma_m(\theta))$. Moreover, for a certain range of θ that includes the special temperature $\theta = \theta_T$, $\hat{\psi}$ is a 'two-well potential'. By this we mean that $\hat{\psi}$ has three local extrema as a function of strain at each fixed θ in this range: two of these are local minima, one in the low-strain phase, one in the high-strain phase, while the third is a local maximum in the intermediate phase; see figure 2. At each of these extrema of $\hat{\psi}$, $(3.2)_1$ shows that the stress vanishes. At $\theta = \theta_T$, both minima carry the same values of $\hat{\psi}$, regardless of the value of λ_T. If $\lambda_T > 0$, the low-strain minimum is lower than the high strain minimum when $\theta > \theta_T$, while the reverse is true if $\theta < \theta_T$. On the other hand, if $\lambda_T < 0$, the relative heights of the two minima are reversed. Thus when $\theta \neq \theta_T$, one minimum is energetically preferred over the other, and the corresponding phase is viewed as energetically stable. At zero stress, both phases are stable in this sense when $\theta = \theta_T$. We speak of θ_T as the *transformation temperature* for the low-to-high strain phase transition, or for the reverse transition. At each θ, the horizontal distance between the low- and high-strain minima is γ_T; we call γ_T the *transformation strain.* The constant λ_T is the *latent heat at the transformation temperature* for the low-strain to high-strain phase transition; it may have either sign.

In addition to (3.6), there are further requirements on the material constants entering (3.5) and (3.7); see Abeyaratne & Knowles (1993) and Abeyaratne & Knowles (1994*b*). These restrictions are

$$\gamma_C = -\frac{\gamma_T}{2} + \frac{M+m}{2}(\theta_C - \theta_T), \quad M+m = 2\alpha - \frac{2\rho_0\lambda_T}{\mu\gamma_T\theta_T}, \quad \gamma_T > (M-m)\theta_C. \tag{3.8}$$

For this particular thermoelastic material, the constitutive statements in (3.2)

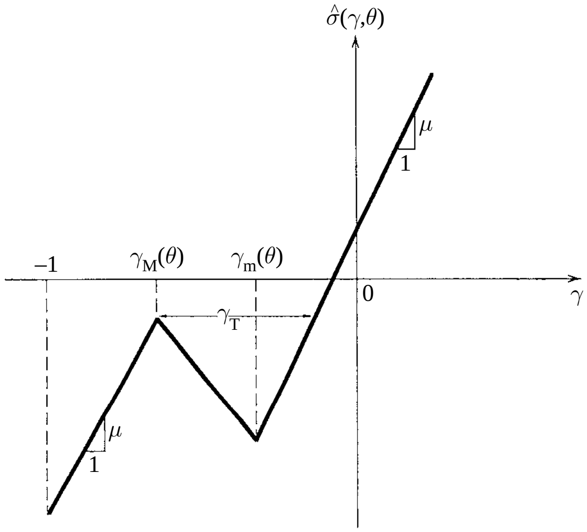

Figure 3. Isothermal stress-strain curve.

specialize as follows:

$$\sigma = \hat{\sigma}(\gamma,\theta) = \begin{cases} \mu\gamma - \alpha\mu(\theta-\theta_{\rm T}) & \text{on } {\rm P}_1, \\ \mu(\gamma+\gamma_{\rm T}) - \alpha\mu(\theta-\theta_{\rm T}) & \text{on } {\rm P}_3, \end{cases} \tag{3.9}$$

$$\eta = \hat{\eta}(\gamma,\theta) = \begin{cases} \dfrac{\alpha\mu}{\rho_0}\gamma + c\left(1+\log\dfrac{\theta}{\theta_{\rm T}}\right) & \text{on } {\rm P}_1, \\ \dfrac{\alpha\mu}{\rho_0}(\gamma+\gamma_{\rm T}) + c\left(1+\log\dfrac{\theta}{\theta_{\rm T}}\right) - \dfrac{\lambda_{\rm T}}{\theta_{\rm T}} & \text{on } {\rm P}_3. \end{cases} \tag{3.10}$$

The relation (3.9) between stress, strain and temperature applies in all thermomechanical processes for which $\theta < \theta_{\rm C}$; if θ is constant, the associated isothermal stress–strain curve is as shown in figure 3.

Stability considerations for nonlinear one-dimensional thermoelasticity in general and for the intermediate phase of the trilinear material in particular are discussed in Abeyaratne & Knowles (1994*a*, *b*).

The ultimate determination of the velocity of a phase boundary in a boundary-initial value problem requires the imposition of a kinetic relation in addition to the boundary- and initial data. For reasons discussed in Abeyaratne & Knowles (1994*b*), we assume here that such a relation has the form

$$\dot{s}(t) = V(f(t)/\langle\theta(t)\rangle), \tag{3.11}$$

where V is a function determined by the material, $\langle\theta(t)\rangle$ is the average of the temperatures

$$\overset{\pm}{\theta}(t)$$

on either side of the phase boundary, and $f(t)$ is the driving traction at the phase boundary at time t. The entropy inequality in the form (2.12) makes it clear that the 'kinetic response function' V of (3.11) must satisfy

$$V(z)z \geqslant 0 \tag{3.12}$$

for all relevant values of its argument $z = f(t)/\langle\theta(t)\rangle$. Although there are many possible forms for V, for reasons of analytical convenience, we shall consider only the simplest possibility: the propagation velocity $\dot{s}$ is proportional to the driving traction f, so that

$$V(z) = \omega z, \tag{3.13}$$

where the material constant $\omega > 0$ is the *mobility* of the phase boundary. It may be noted that the kinetic response function in (3.13) would represent the approximate behaviour of any smooth kinetic relation for small driving traction.

In the impact problem to be discussed in the next section, the bar is assumed to be initially at rest in the low-strain phase at a given temperature θ_0. At time $t = 0$, the particle at the end $x = 0$ of the bar is subject to a prescribed impact velocity $v_0 > 0$, which is maintained for all subsequent time. As we shall see, whether this impact causes the bar to undergo a phase transition will depend on the magnitude of v_0. If a phase change is initiated, we assume that it will occur at the end of the bar, thus originating at the particle whose referential location is $x = 0$. The criterion for the nucleation of a low-strain-to-high-strain phase transition is assumed to be the attainment of a critical value f_* of driving traction f at an incipient phase boundary emerging at $x = 0$; we take f_* to be a materially determined constant.

4. Field equations and jump conditions for the trilinear thermoelastic material

From (3.1), (3.2), one finds that (2.2)–(2.4) specialize as follows for a thermoelastic material:

$$\hat{\sigma}_\gamma(\gamma,\theta)\gamma_x + \hat{\sigma}_\theta(\gamma,\theta)\theta_x = \rho_0 v_t, \tag{4.1}$$

$$v_x = \gamma_t, \tag{4.2}$$

$$\partial\hat{\eta}(\gamma,\theta)/\partial t = 0. \tag{4.3}$$

In general, this is a quasi-linear system of three partial differential equations for the strain γ, the temperature θ and the particle velocity v. Note that (4.3), which is the local version of the first law, says that the specific entropy of a particle is constant where it is smooth. It follows that the *second* law (2.5) is automatically satisfied where η is smooth.

We now wish to specialize the system (4.1)–(4.3) to the trilinear material characterized by (3.7). In doing so, we limit our attention to those processes in the bar for which each particle is always either in the low-strain phase ((γ,θ) in P_1) or in the high-strain phase ((γ,θ) in P_3), i.e. processes in which the intermediate phase ((γ,θ) in P_2) is always absent. Also, we consider only those processes for which $\theta(x,t) < \theta_{\mathrm{C}}$ everywhere in space-time. With these restrictions, (3.9) and (3.10) reduce the system (4.1)–(4.3) to

$$a^2\gamma_x - \alpha a^2\theta_x = v_t, \tag{4.4}$$

$$v_x = \gamma_t, \tag{4.5}$$

$$\alpha a^2\gamma_t + (c/\theta)\theta_t = 0, \tag{4.6}$$

where

$$a = (\mu/\rho_0)^{1/2}. \tag{4.7}$$

The differential equations (4.4)–(4.6) apply *in both the high- and low-strain phases*, provided the fields are smooth.

Let $x = s(t)$ be the location at time t of a strain discontinuity. In specializing the basic jump conditions (2.6)–(2.8) to the trilinear thermoelastic material, one must distinguish two cases: at a *shock wave*, the strains

$$\overset{\pm}{\gamma} = \gamma(s(t)\pm, t)$$

and temperatures

$$\overset{\pm}{\theta} = \theta(s(t)\pm, t)$$

on either side of the jump are such that

$$(\overset{+}{\gamma}, \overset{+}{\theta}) \quad \text{and} \quad (\overline{\gamma}, \overline{\theta})$$

are both in the same phase (either P_1 or P_3); on the other hand, at a *phase boundary*,

$$(\overset{+}{\gamma}, \overset{+}{\theta}) \quad \text{and} \quad (\overline{\gamma}, \overline{\theta})$$

are in *different* phases. We consider a shock wave first. From (3.7), (3.9), (3.10), one finds that (2.6)–(2.8) take the respective forms

$$\left.\begin{aligned} a^2[[\gamma]] - \alpha a^2[[\theta]] + \dot{s}[[v]] = 0, \\ [[v]] + \dot{s}[[\gamma]] = 0, \\ \{[[\theta]] + (\alpha a^2/c)\langle\theta\rangle[[\gamma]]\}\dot{s} = 0, \end{aligned}\right\} \text{ at a shock wave,} \tag{4.8}$$

in either the low-strain or the high-strain phase. From (3.10), the entropy inequality (2.9) is found to be

$$\{\alpha a^2[[\gamma]] + c\log(\overset{+}{\theta}/\overline{\theta})\}\dot{s} \leqslant 0 \text{ at a shock wave.} \tag{4.9}$$

If (4.8) is considered as a homogeneous linear system for $[[\gamma]]$, $[[\theta]]$ and $[[v]]$, and if not all of these jumps vanish, then the determinant of the system must be zero, and the shock wave velocity $\dot{s}$ must satisfy

$$\dot{s}\left\{\dot{s}^2 - a^2\left(1 + \frac{\alpha^2 a^2}{c}\langle\theta\rangle\right)\right\} = 0. \tag{4.10}$$

One root of (4.10) is $\dot{s} = 0$, corresponding to a shock wave that is stationary in the Lagrangian sense. The Eulerian image of this discontinuity is of course moving, but it always consists of the same material particles; it is thus the counterpart of a *contact discontinuity* in gas dynamics. Contact discontinuities occur in the Riemann problem analysed in Abeyaratne & Knowles (1994*b*), but they do not arise in the impact problem considered here. The remaining two roots of (4.10) correspond to the Lagrangian velocity of genuine shock waves, expressed in terms of the average $\langle\theta\rangle$ of the temperatures on either side of the shock:

$$\dot{s} = \pm a\left(1 + \frac{\alpha^2 a^2}{c}\langle\theta\rangle\right)^{1/2}. \tag{4.11}$$

Next, suppose that the strain discontinuity is a phase boundary; for our purposes, phase boundaries of interest will always have the high-strain phase on the left, the low-strain phase on the right. Using (3.7), (3.9) and (3.10) in (2.6)–(2.8) gives the jump conditions at a phase boundary in a trilinear thermoelastic material:

$$\left.\begin{aligned} a^2([[\gamma]] - \gamma_{\mathrm{T}}) - \alpha a^2[[\theta]] + \dot{s}[[v]] = 0,\\ [[v]] + \dot{s}[[\gamma]] = 0,\\ \{-\gamma_{\mathrm{T}} a^2(\langle\gamma\rangle + \gamma_{\mathrm{T}}/2) + \alpha a^2\langle\theta\rangle[[\gamma]]\\ +c[[\theta]] + \lambda_{\mathrm{T}} - \alpha a^2\gamma_{\mathrm{T}}\theta_{\mathrm{T}}\}\dot{s} = 0, \end{aligned}\right\} \text{ at a phase boundary.} \tag{4.12}$$

The entropy inequality (2.9) becomes

$$\{\alpha a^2[[\gamma]] + c\log(\overset{+}{\theta}/\overline{\theta}) - \alpha a^2\gamma_{\mathrm{T}} + \lambda_{\mathrm{T}}/\theta_{\mathrm{T}}\}\dot{s} \leqslant 0 \text{ at a phase boundary.} \tag{4.13}$$

It may be noted that if one formally sets the transformation strain γ_{T} and the latent heat λ_{T} equal to zero, the jump conditions (4.12) and the entropy inequality (4.13) for a phase boundary reduce formally to their respective counterparts (4.8) and (4.9) for shock waves.

For the trilinear material, the driving traction at a phase boundary with the high-strain phase on the left can be calculated from (3.2), (3.9), (3.10) and (2.10); after simplifying the result with the help of the energy jump condition $(4.12)_3$, one finds

$$f = \rho_0\langle\theta\rangle\{\alpha a^2(\gamma_{\mathrm{T}} - [[\gamma]]) - c\log(\overset{+}{\theta}/\overline{\theta}) - \lambda_{\mathrm{T}}/\theta_{\mathrm{T}}\}. \tag{4.14}$$

5. The impact problem

We seek a piecewise smooth solution $\gamma(x,t)$, $v(x,t)$, $\theta(x,t)$ of the field equations (4.1)–(4.3) in the first quadrant of the x,t-plane for the particular thermoelastic material governed by the potential (3.7) subject to the following initial and boundary conditions:

$$\gamma(x,0) = v(x,0) = 0, \qquad \theta(x,0) = \theta_0 \quad \text{for} \quad x > 0, \tag{5.1}$$

$$v(0,t) = v_0 \quad \text{for} \quad t > 0, \tag{5.2}$$

where $\theta_0 > 0$ and v_0 are given constants. Although v_0 may have either sign, the case $v_0 > 0$ corresponding to impact will be of principal interest. The initial temperature θ_0 is assumed less than θ_{C}, and the point in the γ,θ-plane with coordinates $(0,\theta_0)$, corresponding to the state of the bar before impact, is assumed to be in the region P_1 corresponding to the low-strain phase.

In a purely mechanical theory that ignores thermal effects, as in Abeyaratne & Knowles (1991), or when such effects are accounted for along with heat conduction, as in Abeyaratne & Knowles (1994*a*), one can prove that the intermediate phase corresponding to the region P_2, if absent initially, is always absent, at least in certain initial value problems. In the present adiabatic framework, we have been unable to establish analogous results, so we legislate the absence of the intermediate phase *a priori*: attention is limited to processes for which, at each instant, a given particle is either in the low-strain phase $((\gamma(x,t),\theta(x,t))$ in $\mathrm{P}_1)$ or in the high-strain phase $((\gamma(x,t),\theta(x,t))$ in $\mathrm{P}_3)$. It is also assumed that $\theta(x,t) < \theta_{\mathrm{C}}$ everywhere in space-time. Thus (3.7), (3.9) and (3.10) apply, and the field equations (4.1)–(4.3) specialize to (4.4)–(4.6) wherever the fields are smooth.

It is also necessary that, at any shock wave or phase boundary arising in a solution of (4.4)–(4.6), the jump conditions (4.8) or (4.12) hold, respectively, and that the en-

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tropy inequalities (4.9) and (4.13) are satisfied at shock waves and phase boundaries, respectively.

The boundary-initial value problem is invariant under the scale change $x \to kx$, $t \to kt$ for any constant k, suggesting that we seek solutions with this same invariance. This leads to the ansatz that $\gamma = \gamma(x/t)$, $v = v(x/t)$, $\theta = \theta(x/t)$ where the fields are smooth. With this ansatz in force, the differential equations (4.4)–(4.6) reduce to a set of homogeneous linear algebraic equations for $\gamma'(\xi)$, $v'(\xi)$ and $\theta'(\xi)$, where $\xi = x/t$:

$$\left.\begin{aligned} a^2\gamma'(\xi) - \alpha a^2\theta'(\xi) + \xi v'(\xi) &= 0,\\ \xi\gamma'(\xi) + v'(\xi) &= 0,\\ \alpha a^2\gamma'(\xi) + c\theta'(\xi)/\theta(\xi) &= 0. \end{aligned}\right\} \tag{5.3}$$

To understand the implications of (5.3), it is helpful to distinguish two cases.

Case 1. The coefficient of thermal expansion vanishes: $\alpha = 0$. In this case, the determinant of the system (5.3) vanishes if and only if $\xi \equiv x/t = \pm a$. Moreover, according to (4.11), a is the speed of shock waves when $\alpha = 0$. Thus where the fields are smooth, γ, θ and v must be constant in this case. In Case 1, we therefore seek piecewise constant solutions of the impact problem.

Case 2. The coefficient of thermal expansion does not vanish: $\alpha > 0$. In this case, the determinant of the system (5.3) vanishes if and only if

$$\theta(\xi) = \frac{c}{\alpha^2 a^4}(\xi^2 - a^2); \tag{5.4}$$

since $\theta > 0$, necessarily $\xi \equiv x/t > a$ in Case 2. Any connected subregion of the first quadrant of the x, t-plane in which (5.4) holds must be a sector $\xi_1 < x/t < \xi_2$ with vertex at the origin, and, by (5.3), $\gamma(\xi)$ and $v(\xi)$ must take the following forms in such a sector:

$$\gamma(\xi) = -\frac{c}{\alpha a^2}\log(\xi^2 - a^2) + A, \qquad v(\xi) = \frac{2c}{\alpha a^2}\left\{\xi + \frac{a}{2}\log\left(\frac{\xi - a}{\xi + a}\right)\right\} + B, \tag{5.5}$$

where A and B are arbitrary constants. In any region of the x, t-plane where the fields are smooth and (5.4) *fails* to hold, γ, v and θ must be constant. Thus in Case 2, we seek solutions of the impact problem in which, in every sector $\xi_1 < x/t < \xi_2$ of the first quadrant of the x, t-plane in which the fields are smooth, γ, v and θ are either constant or given by the *fan* (5.4), (5.5).

In the next section, we consider Case 1 in detail. Although $\alpha = 0$ is not a realistic case physically, it describes the main qualitative features of the impact problem for the trilinear thermoelastic material. In § 7, we address briefly the modifications of the results of § 6 necessitated by non-zero values of α by describing approximations appropriate for small values of the Grüneisen parameter ε, which in the present circumstances is defined by

$$\varepsilon = \alpha a^2/c = \hat{\psi}_{\gamma\theta}/(\theta\hat{\psi}_{\theta\theta}). \tag{5.6}$$

As defined in (5.6), ε corresponds to the *modified Grüneisen parameter* employed by Clifton in eqn (20) of Clifton (1993). In a general thermoelastic material, ε will depend on the strain and the temperature; for the trilinear special case considered here, ε is a constant. Values of ε near 1 are consistent with some experiments on certain ceramics (see Tan & Ahrens 1990).

6. Solutions of the impact problem when $\alpha = 0$

(a) Construction of solutions

In this section, we consider the special case $\alpha = 0$. The differential equations (4.4)–(4.6) then simplify to

$$a^2\gamma_x = v_t, \qquad v_x = \gamma_t, \tag{6.1}$$

$$\theta_t = 0. \tag{6.2}$$

Thus mechanical and thermal effects are uncoupled in the differential equations, and the mechanical equations (6.1) are equivalent to the scalar wave equation with wave speed $a = (\mu/\rho_0)^{1/2}$. At a shock wave, the jump conditions (4.8) and the entropy inequality (4.9) with $\alpha = 0$ become

$$\left.\begin{aligned} a^2[[\gamma]] + \dot{s}[[v]] = 0 \\ [[v]] + \dot{s}[[\gamma]] = 0 \\ [[\theta]]\dot{s} = 0 \end{aligned}\right\} \quad \text{(shock wave, } \alpha = 0\text{)}, \tag{6.3}$$

$$\{\log(\overset{+}{\theta}/\overline{\theta})\}\dot{s} \leqslant 0 \quad \text{(shock wave, } \alpha = 0\text{)}, \tag{6.4}$$

respectively. From $(6.3)_{1,2}$, one finds that a shock wave travels with speed $\dot{s} = a$, so that from $(6.3)_3$, the temperature remains continuous across a shock wave, and the entropy inequality (6.4) is trivially satisfied.

At a phase boundary, the jump conditions (4.12) and the entropy inequality (4.13) reduce when $\alpha = 0$ to

$$\left.\begin{aligned} a^2[[\gamma]] + \dot{s}[[v]] - \gamma_{\mathrm{T}}a^2 = 0 \\ [[v]] + \dot{s}[[\gamma]] = 0 \\ \{c[[\theta]] - \gamma_{\mathrm{T}}a^2[\langle\gamma\rangle + \gamma_{\mathrm{T}}/2] + \lambda_{\mathrm{T}}\}\dot{s} = 0 \end{aligned}\right\} \quad \text{(phase boundary, } \alpha = 0\text{)}, \tag{6.5}$$

and

$$\{(\lambda_{\mathrm{T}}/\theta_{\mathrm{T}}) + c\log(\overset{+}{\theta}/\overline{\theta})\}\dot{s} \leqslant 0 \quad \text{(phase boundary, } \alpha = 0\text{)}, \tag{6.6}$$

respectively.

If the impact fails to cause a phase change, we expect that the only propagating discontinuity will be a shock wave at $x = at$, the temperature remaining continuous everywhere. Recalling that scale-invariant solutions are piecewise constant when $\alpha = 0$, we conclude from (6.1), (6.2), (5.1), (5.2) and (6.3) that the appropriate solution in the absence of a phase transition is given by

$$\gamma, v, \theta = \begin{cases} -v_0/a, & v_0, \quad \theta_0 \quad \text{for } 0 \leqslant x < at, \\ \quad 0, & \;0, \quad \theta_0 \quad \text{for } x > at; \end{cases} \tag{6.7}$$

this solution is described graphically in the x, t-plane in figure 4*a*.

On the other hand, if there is a phase change, the appropriate solution will involve at propagating phase boundary. If the phase boundary were the *front-running* disturbance in this solution, then $(6.5)_{1,2}$ would show that $\dot{s} < a$, so there would be no shock wave. Using the jump conditions, boundary conditions and initial conditions, one can then show that there is no solution involving a front-running phase boundary that satisfies the impact condition $v(0,t) = v_0$. Hence the phase boundary must be

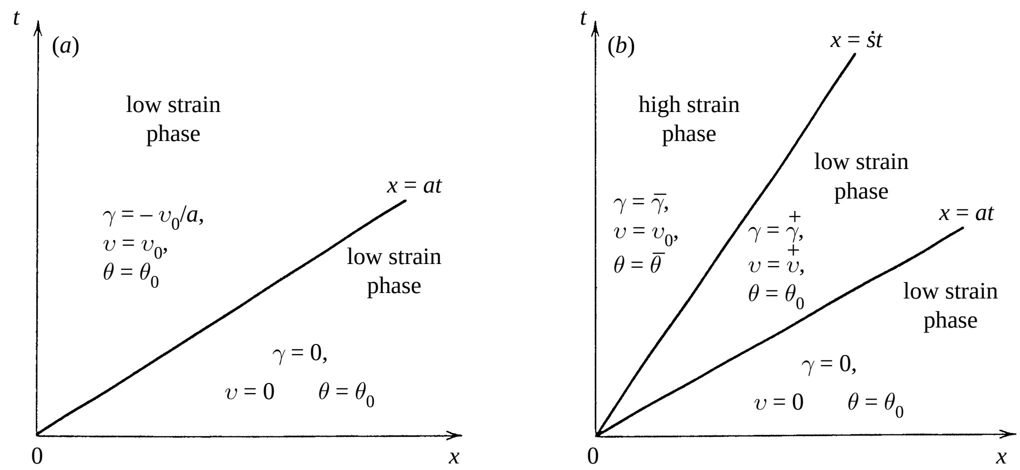

Figure 4. The x, t-plane for the impact problem in the cas $\alpha = 0$: (*a*) without phase transition; (*b*) with phase transition.

preceded by a shock wave, so that $0 < \dot{s} < a$, suggesting a solution of the following form when a phase change occurs:

$$\gamma, v, \theta = \begin{cases} \overline{\gamma}, & v_0, & \overline{\theta} & \text{for } 0 \leqslant x < \dot{s}t, \\ \overset{+}{\gamma}, & \overset{+}{v}, & \theta_0 & \text{for } \dot{s}t < x < at, \\ 0, & 0, & \theta_0 & \text{for } x > at; \end{cases} \tag{6.8}$$

see figure 4*b*. Here $\dot{s}$ is the as yet undetermined constant phase boundary velocity, and $\overset{\pm}{\gamma}$, $\overline{\theta}$, $\overset{+}{v}$ are unknown constants. On enforcing the jump conditions (6.3) at the shock wave $x = at$ and (6.5) at the phase boundary $x = \dot{s}t$, one finds after some algebra that

$$\overline{\gamma} = -\frac{v_0}{a} - \gamma_T \frac{a}{a + \dot{s}}, \tag{6.9}$$

$$\overline{\theta} = \theta_0 + \frac{\lambda_T}{c} + \frac{a\gamma_T}{c} v_0 + \frac{a^2 \gamma_T^2}{2c} \frac{\dot{s}^2 - 2a\dot{s}}{a^2 - \dot{s}^2}, \tag{6.10}$$

$$\overset{+}{\gamma} = -\frac{v_0}{a} + \gamma_T \frac{a\dot{s}}{a^2 - \dot{s}^2}, \tag{6.11}$$

$$\overset{+}{v} = v_0 - \gamma_T \frac{a^2 \dot{s}}{a^2 - \dot{s}^2}; \tag{6.12}$$

equations (6.9)–(6.12) determine all unknowns in terms of $\dot{s}$. Further, the entropy inequality (6.6) now requires that

$$\frac{\lambda_T}{c\theta_T} - \log\left\{1 + \frac{\lambda_T}{c\theta_0} + \frac{a^2\gamma_T}{2c\theta_0}\left[2\frac{v_0}{a} + \gamma_T \frac{\dot{s}^2 - 2a\dot{s}}{a^2 - \dot{s}^2}\right]\right\} \leqslant 0. \tag{6.13}$$

(*b*) *Entropy and phase segregation inequalities*

In order that either the no-phase-change solution (6.7) or the phase-change solution (6.8)–(6.12) be valid, each strain–temperature pair arising in these solutions must correspond to a point in the appropriate region P_i of the γ, θ-plane that corresponds

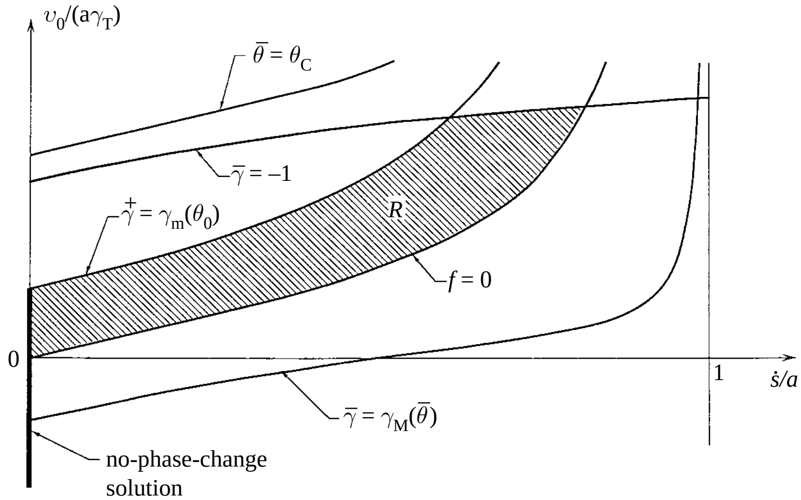

Figure 5. Region R in the $\dot{s}/a$, $v_0/(a\gamma_T)$-plane permitted by phase segregation and entropy inequalities: $\lambda_T = 0$, $\alpha = 0$.

to the underlying phase. In particular, since the bar is assumed to be in the low-strain phase initially, the point $(0, \theta_0)$ must lie in the low-strain region P_1, so that the initial datum θ_0 must be such that $\gamma_m(\theta_0) < 0$. Next, considering the no-phase-change solution, we observe that the point $(-v_0/a, \theta_0)$ must lie in the region P_1, since the material behind the shock wave remains in the low-strain phase. Thus for a given initial temperature θ_0 satisfying $\gamma_m(\theta_0) < 0$, the impact velocity v_0 must be small enough to assure that $v_0/a < -\gamma_m(\theta_0)$.

In the phase-change solution (6.8)–(6.12), the data θ_0, v_0 *and the phase boundary velocity* $\dot{s}$ must be such that, in addition to the entropy inequality (6.13), the restrictions

$$\bar{\gamma} > -1, \quad \bar{\gamma} < \gamma_M(\bar{\theta}), \quad \overset{+}{\gamma} > \gamma_m(\theta_0), \quad \bar{\theta} < \theta_C$$

must also hold. Regarding the initial temperature θ_0 as given, fixed and satisfying $\gamma_m(\theta_0) < 0$, these inequalities may be conveniently described in the plane in which the dimensionless velocities $\dot{s}/a$ and $v_0/(a\gamma_T)$ are the Cartesian coordinates. Each of the five inequalities cited above corresponds to a requirement that the point $(\dot{s}/a, v_0/(a\gamma_T))$ be either below or above a certain curve in this plane. To illustrate the nature of these curves and the associated region in the $\dot{s}/a, v_0/(a\gamma_T)$-plane permitted by the five inequalities, we refer to figure 5. For definiteness, this figure is drawn for the special case of zero latent heat: $\lambda_T = 0$; this case arises physically when the two 'phases' of the material are actually variants of a single phase, as in the twinning of crystals. In this special case, all of the restrictions on θ_0, v_0 and $\dot{s}$ cited above may be expressed in a form that involves only the dimensionless material constants $N = Ma^2\gamma_T/c$, $T_C = c\theta_C/(a^2\gamma_T^2)$ and γ_T as well as the dimensionless initial temperature $T_0 = c\theta_0/(a^2\gamma_T^2)$. Figure 5 corresponds to the choice $N = 0.1$, $T_C = 1.25$, $\gamma_T = 0.5$, $T_0 = 0.1$; these values of the dimensionless parameters are chosen for reasons of graphical clarity and do not necessarily correspond to the characteristics of any specific real material. The shaded region R in the figure represents the set of all permissible pairs $\dot{s}$, v_0 for which the solution (6.8)–(6.12) involving a phase change is valid.

The portion of the vertical axis shown bold in figure 5 corresponds to the inequality $v_0/a < -\gamma_m(\theta_0)$, i.e. to the set of values of the impact velocity v_0 for which, at the given initial temperature θ_0, the solution (6.7) without a phase change holds.

The phase-change solution (6.8)–(6.12) satisfies all jump conditions, differential equations, boundary and initial conditions, regardless of the value of $\dot{s}$, subject only to the requirement $0 < \dot{s} < a$, the entropy inequality (6.13) and the phase segregation inequalities cited above and exemplified in figure 5. Thus further information is required to determine $\dot{s}$, once it has been ascertained that the phase-change solution, and not the no-phase-change solution (6.7), is appropriate. This determination of $\dot{s}$ is accomplished with the help of a kinetic relation; the question of which of the solutions (6.7) or (6.8)–(6.12) is the proper one is then addressed by appealing to a nucleation criterion.

(*c*) *Kinetics and nucleation*

According to (3.11), (3.13), the kinetic relation to be used here to describe the evolution of the phase transition is

$$\dot{s} = \omega f/\langle\theta\rangle, \tag{6.14}$$

where f is the driving traction at the phase boundary. By using (4.14) and (6.8)–(6.11), one finds that (6.14) takes the form

$$\dot{s} = -\frac{\omega\rho_0\lambda_{\mathrm{T}}}{\theta_{\mathrm{T}}} + \omega\rho_0 c \log\left(1 + \frac{\lambda_{\mathrm{T}}}{c\theta_0} + \frac{a\gamma_{\mathrm{T}}}{c\theta_0}v_0 + \frac{a^2\gamma_{\mathrm{T}}^2}{2c\theta_0}\frac{\dot{s}^2 - 2a\dot{s}}{a^2 - \dot{s}^2}\right). \tag{6.15}$$

This relation between the impact velocity and the phase boundary velocity can be represented by a curve in the $\dot{s}/a, v_0/(a\gamma_{\mathrm{T}})$-plane. For the special case $\lambda_{\mathrm{T}} = 0$ of zero latent heat, this 'kinetic curve' K is shown in relation to the region R permitted by the entropy and phase segregation inequalities in figure 6. The figure is drawn for the same values of the dimensionless parameters N, γ_{T}, T_{C} and T_0 underlying figure 5; in addition the dimensionless mobility $\Omega = \omega c\rho_0/a$ has been given the value 0.4. For any given positive values of the impact velocity v_0 and the initial temperature, (6.15) determines a unique value of $\dot{s}$ between zero and a, as is clear from figure 6, at least in the special case described by the figure. We write

$$\dot{s}/a = \varphi(v_0, \theta_0) \tag{6.16}$$

for the dimensionless value of the phase boundary velocity determined in this way. If the corresponding point $(\dot{s}/a, v_0/(a\gamma_{\mathrm{T}})) \in K$ lies in the shaded admissible region R, then it singles out the unique phase-change solution (6.8)–(6.12) conforming to the kinetic relation. It should be noted that, for sufficiently large values of the impact velocity v_0, the kinetic curve leads to solutions (6.8) that violate the requirement $\gamma > -1$ and hence are unacceptable.

From figure 6, it is clear that if the impact velocity v_0 is negative (causing tension rather than compression), no phase change can occur according to the present material model. On the other hand, if v_0 is positive and large enough, a phase change must occur. For an intermediate range of positive values of v_0, the bar may or may not undergo a phase change; see figure 6. To determine which alternative occurs for impact velocities in this intermediate range, we impose the nucleation criterion. This criterion asserts that the particle at the point of impact will change phase, causing a phase boundary to emerge at $x = 0$ and move into the bar, if the driving traction f caused by doing so is at least as great as a certain critical value f_*; we take f_*

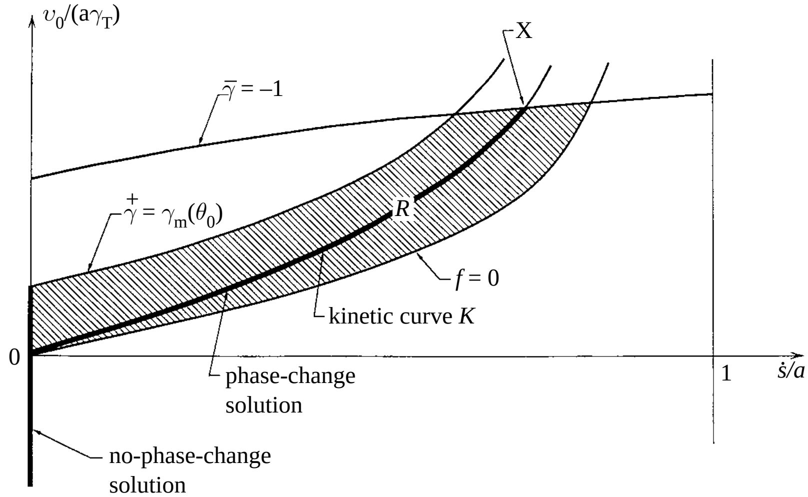

Figure 6. The kinetic curve K in the region R: $\lambda_T = 0$, $\alpha = 0$.

to be a material constant. From (4.14) and (6.8)–(6.11), one finds that the driving traction f is given by

$$f = c\rho_0\theta_0[-\lambda_T/(c\theta_T) + \log(1+Q)](1 + \tfrac{1}{2}Q), \tag{6.17}$$

where

$$Q = Q(\dot{s}, v_0) \equiv \frac{\lambda_T}{c\theta_0} + \frac{a\gamma_T}{c\theta_0}v_0 + \frac{a^2\gamma_T^2}{2c\theta_0}\frac{\dot{s}^2 - 2a\dot{s}}{a^2 - \dot{s}^2}. \tag{6.18}$$

If $F_* = f_*/(c\rho_0\theta_C)$, the nucleation criterion requires that

$$(1 + Q/2)[-\lambda_T/(c\theta_T) + \log(1+Q)] \geqslant F_*\theta_C/\theta_0. \tag{6.19}$$

For a given initial temperature θ_0, the nucleation inequality (6.19) will hold only on a subregion of the region R in the $\dot{s}/a, v_0/(a\gamma_T)$-plane. Only those two-phase solutions corresponding to points in this subregion are then available, and therefore only that portion K_{nuc} of the kinetic curve K that lies in this subregion is relevant. For the special case $\lambda_T = 0$, K_{nuc} is shown bold in figure 7. The figure is drawn for the same values of the dimensionless parameters used for figures 5 and 6; in addition, the special value $F_* = 0.8$ of the dimensionless critical driving traction was assumed. If the imposed particle velocity v_0 exceeds the critical value $v_*(\theta_0)$ defined in figure 7, then the phase-change solution is the appropriate one, and the value $\dot{s}$ of the phase boundary velocity is that determined by the kinetic curve for the given v_0. For values of v_0 less than $v_*(\theta_0)$, the bar does not undergo a phase change, and the solution (6.7) applies.

(*d*) *Results*

Here we describe some features of the two types of solution constructed above; we assume that the impact velocity is positive, corresponding to compression.

In typical flyer-plate or projectile-impact experiments, the specimen is a plate subjected to impact on one face, and the particle velocity time-history is measured at a point on the opposite face. In such experiments, there are of course repeated wave reflections from the faces and the lateral surface of the specimen. Reflections are absent in the present analysis, because we have chosen the bar to be semi-infinite. It

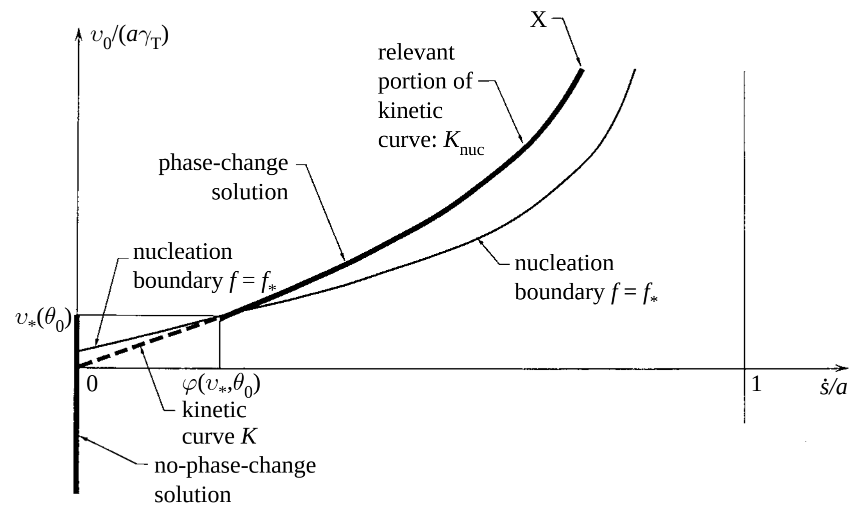

Figure 7. Effect of nucleation criterion and kinetic relation on the selection of solution: $\lambda_T = 0$, $\alpha = 0$.

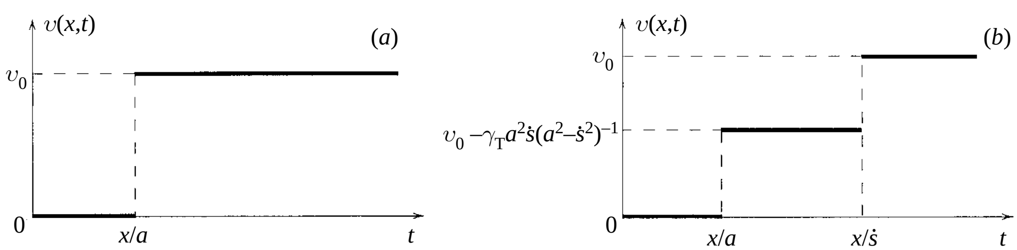

Figure 8. Schematic plot of the velocity time-history of a particle in the case $\alpha = 0$: (*a*) without phase change; (*b*) with phase change.

is nevertheless of interest to compare the time-histories of particle velocity $v(x,t)$ at a fixed particle in the no-phase-change solution (6.7) and in the solution (6.8)–(6.12) with a phase change. By (6.7), in the absence of a phase transition,

$$v(x,t) = \begin{cases} 0, & t < x/a, \\ v_0, & t > x/a; \end{cases} \tag{6.20}$$

the corresponding graph is shown in figure 8*a*. When a phase change does occur, (6.8) and (6.12) give

$$v(x,t) = \begin{cases} 0 & \text{for } t < x/a, \\ v_0 - a\gamma_T \dfrac{\dot{s}/a}{1-(\dot{s}/a)^2} & \text{for } x/a < t < x/\dot{s}, \\ v_0 & \text{for } t > x/\dot{s}. \end{cases} \tag{6.21}$$

The phase boundary velocity $\dot{s}$ entering (6.21) is determined in terms of the impact velocity v_0 and the initial temperature θ_0 by the kinetic relation (6.15): $\dot{s}/a = \varphi(v_0, \theta_0)$. The value of $\dot{s}$ returned by this calculation is necessarily between zero and a, so that we may schematically describe the time history of $v(x,t)$ when a phase change takes place as in figure 8*b*. Observe that the first discontinuity in v always leads to a particle velocity less than the final velocity v_0, which is achieved after the

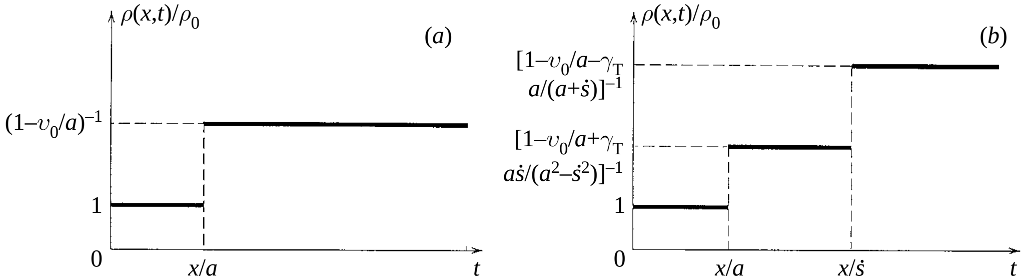

Figure 9. Schematic plot of the density time-history of a particle in the case $\alpha = 0$: (a) without phase change; (b) with phase change.

passage of the phase boundary. The disparity between the intermediate and final values of particle velocity increases with the transformation strain γ_{T} and with the phase boundary velocity $\dot{s}$. Indeed, if the transformation strain is sufficiently large or the phase boundary travels fast enough, the intermediate and final particle velocities may have opposite signs.

If the deformation of the one-dimensional continuum under consideration is viewed as uniaxial, mass balance requires that the initial density ρ_0 and the current density $\rho(x,t)$ at time t at the particle whose reference location is x are related by

$$\rho(x,t) = \frac{\rho_0}{1+\gamma(x,t)}, \tag{6.22}$$

where $\gamma(x,t)$ is the strain at this particle at time t. Using either (6.7) or (6.8), (6.9) and (6.11), we can describe the time-history of density at a given particle. When there is no phase change,

$$\rho(x,t)/\rho_0 = \begin{cases} 1 & \text{for } t < x/a, \\ \dfrac{1}{1-v_0/a} & \text{for } t > x/a; \end{cases} \tag{6.23}$$

this no-phase-change density ratio is shown schematically in figure 9a.

When there is a phase change, (6.23) is replaced by

$$\rho(x,t)/\rho_0 = \begin{cases} 1 & \text{for } t < x/a, \\ \dfrac{1}{1-v_0/a+\gamma_{\mathrm{T}}(\dot{s}/a)/[1-(\dot{s}^2/a^2)]} & \text{for } x/a < t < x/\dot{s}, \\ \dfrac{1}{1-v_0/a-\gamma_{\mathrm{T}}/(1+\dot{s}/a)} & \text{for } t > x/\dot{s}; \end{cases} \tag{6.24}$$

the schematic graph associated with (6.24) is shown in figure 9b.

When the bar undergoes a phase transition, (6.24) shows that – according to the present model – the final density ρ_∞ is given as a function of impact velocity and initial temperature by

$$\rho_\infty = \rho_\infty(v_0,\theta_0) = \frac{\rho_0}{1-v_0/a-\gamma_{\mathrm{T}}/[1+\varphi(v_0,\theta_0)]}, \tag{6.25}$$

where $\dot{s}/a = \varphi(v_0,\theta_0)$ is the value of the dimensionless phase boundary velocity determined by the kinetic relation (6.15). Since $\varphi(v_0,\theta_0) > 0$, it follows from (6.24)

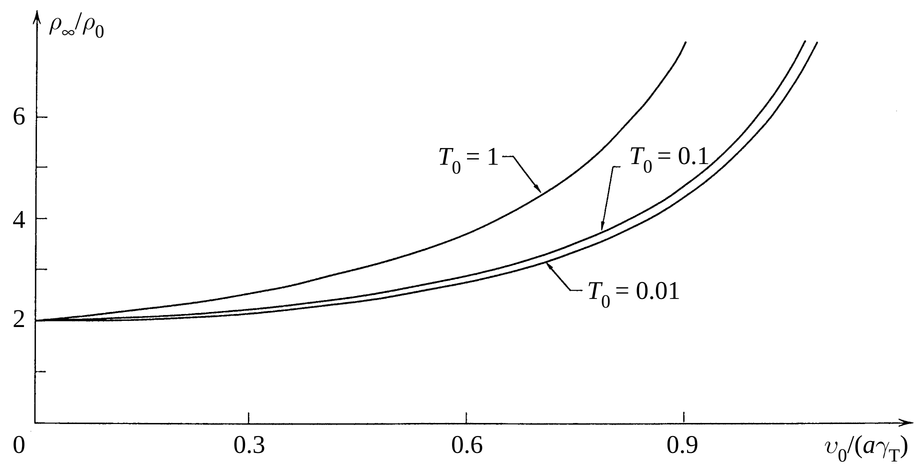

Figure 10. Ratio of final to initial density after a phase change as a funtion of impact velocity at various initial temperatures: $\lambda_T = 0$, $\alpha = 0$.

and (6.23) that the final density in the presence of a phase transition always exceeds the final density when no transition occurs. Moreover, referring to figure 7, we observe that, as the point X of intersection between the kinetic curve and the curve $\overline{\gamma} = -1$ is approached from below, the strain behind the phase boundary tends to -1, so that by (6.22), the final density tends to infinity in this limit. This will occur at a limiting value of the impact velocity v_0 that depends on the initial temperature, and of course on the material parameters as well.

From (6.24) and (6.15), one sees that, in addition to the impact velocity and initial temperature, the final density also depends parametrically on the shock wave speed $a = (\mu/\rho_0)^{1/2}$, the transformation strain γ_T, the latent heat λ_T, the specific heat at constant strain c, the reference density ρ_0 and the phase boundary mobility ω. When the latent heat λ_T vanishes, it can be shown that ρ_∞/ρ_0 depends only on the dimensionless initial temperature $T_0 = c\theta_0/(a^2\gamma_T^2)$, the dimensionless impact velocity $v_0/(a\gamma_T)$ and the dimensionless mobility $\Omega = \omega c\rho_0/a$. In figure 10, the ratio ρ_∞/ρ_0 of final to initial densities for the impact-induced phase transition is plotted in the case $\lambda_T = 0$ as a function of dimensionless impact velocity $v_0/(a\gamma_T)$ for various dimensionless initial temperatures T_0, a fixed value $\Omega = 0.4$ being assumed for the dimensionless mobility. The figure does not reflect the effect of nucleation: only that portion of each curve lying to the right of the critical impact velocity $v_*(\theta_0)$ is relevant. Observe that the largest and smallest values of T_0 involved in figure 10 differ by two orders of magnitude, suggesting that the final density during the phase change is rather insensitive to the initial temperature, at least at this mobility and when the coefficient of thermal expansion $\alpha = 0$.

It is an artifact of the present case $\alpha = 0$ that the temperature does not jump across a shock wave, so that – as in (6.7) – the final temperature and the initial temperature coincide when the impact fails to cause a phase transition. When there is a phase change, (6.10) and the kinetic relation in the form $\dot{s}/a = \varphi(v_0, \theta_0)$ as in (6.16) provide the final temperature $\theta_\infty = \overline{\theta}$ in terms of the impact velocity v_0 and the initial temperature θ_0 and, of course, material constants. In the special case of zero latent heat, we plot in figure 11 the ratio θ_∞/θ_0 as a function of the dimensionless impact velocity $v_0/(a\gamma_T)$ at various values of the dimensionless initial temperature

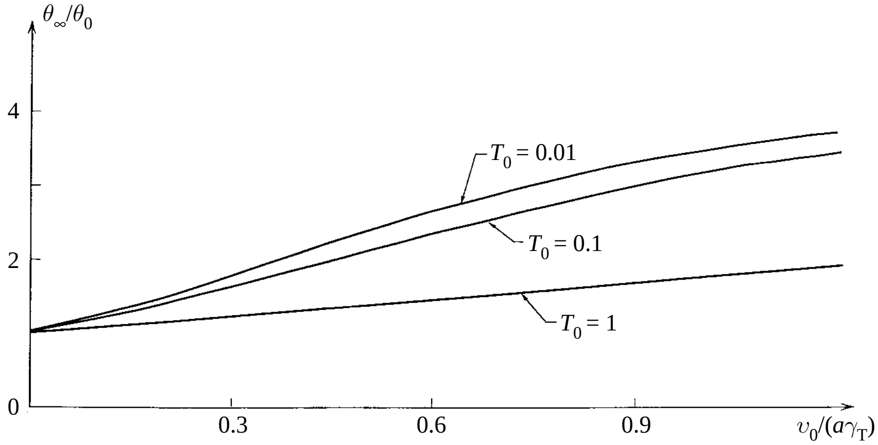

Figure 11. Ratio of final to initial temperature after a phase change as a funtion of impact velocity at various initial temperatures: $\lambda_{\mathrm{T}} = 0$, $\alpha = 0$.

T_0; the effect of nucleation is not shown in the figure. Observe that, at a given impact velocity in this model, θ_∞/θ_0 *decreases* as T_0 increases.

7. Solutions of the impact problem when $\alpha \neq 0$

According to the discussion in § 5, when the coefficient of thermal expansion α does not vanish, scale-invariant solutions of the impact problem may involve fans of the form (5.4), (5.5) as well as constant states. In the present section, we describe the qualitative structure of such solutions when $\alpha \neq 0$, and we give quantitative approximations for some of the resulting fields when the Grüneisen parameter $\varepsilon = \alpha a^2/c$ is small. It is convenient for these purposes to introduce a dimensionless temperature T and a dimensionless particle velocity w by setting

$$T = c\theta/(a^2\gamma_{\mathrm{T}}^2), \qquad w = v/(a\gamma_{\mathrm{T}}); \tag{7.1}$$

see the discussion between (6.13) and (6.14), where these same non-dimensional versions of temperature and particle velocity were also used. The dimensionless versions of the initial temperature θ_0, the transformation temperature θ_{T} and the impact velocity v_0 are denoted by T_0, T_{T} and w_0, respectively.

As in the case $\alpha = 0$, there are two fundamental types of solutions: one in which the bar does not undergo a phase transition, and the other in which it does. We discuss the form of the no-phase-change solutions first.

(a) *Solutions without a phase-change*

We begin by asking whether there are solutions which involve a *shock wave, but no fans and no phase boundary*, corresponding to the structure indicated in figure 12*a*. For a solution of this form, the jump conditions (4.8) appropriate to a shock wave show that the strain $\overline{\gamma}$ and dimensionless temperature $\overline{T}$ behind the shock are given in terms of the dimensionless shock wave velocity $\xi_1 = \dot{s}_1/a$ by

$$\overline{\gamma} = -\gamma_{\mathrm{T}} w_0/\xi_1, \qquad \overline{T} = T_0 \frac{1 + \varepsilon\gamma_{\mathrm{T}} w_0/(2\xi_1)}{1 - \varepsilon\gamma_{\mathrm{T}} w_0/(2\xi_1)}, \tag{7.2}$$

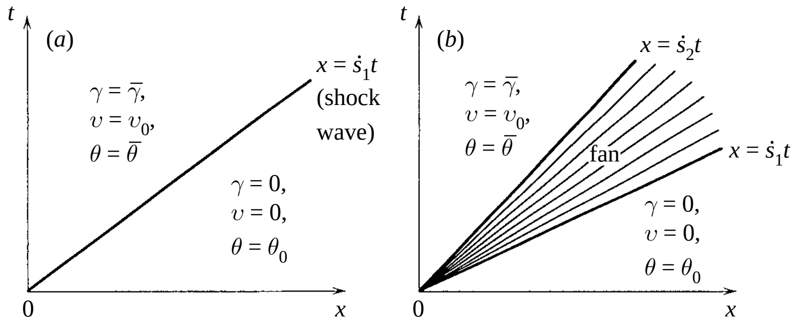

Figure 12. The x,t-plane for the impact problem in the case $\alpha \neq 0$ without a phase transition: (a) with shock wave; (b) with fan.

and that ξ_1 satisfies

$$\xi_1^2 = 1 + \frac{\varepsilon^2 \gamma_T^2 T_0}{1 - \varepsilon \gamma_T w_0/(2\xi_1)}. \tag{7.3}$$

For small ε, (7.3) and (7.2) give

$$\left.\begin{aligned} \xi_1 &= 1 + \gamma_T^2 T_0 \varepsilon^2/2 + O(\varepsilon^3), \\ \bar{\gamma} &= -\gamma_T w_0 + w_0 T_0 \gamma_T^3 \varepsilon^2/2 + O(\varepsilon^3), \\ \bar{T} &= T_0 + \gamma_T w_0 T_0 \varepsilon + O(\varepsilon^2). \end{aligned}\right\} \tag{7.4}$$

The entropy inequality (4.9) at a shock wave can be shown to require that $0 \leqslant \gamma_T w_0/\xi_1 \leqslant 2/\varepsilon$; in particular, no entropically admissible solution without a phase change and without a fan exists unless the impact velocity v_0 is positive, corresponding to compression of the bar. When v_0 is positive, $(7.2)_2$ shows that the temperature increases behind the shock wave.

Next, we consider the possibility that a no-phase-change solution involves a *fan, but no shock wave*; thus we seek solutions of the form indicated in figure 12b, where, within the fan, the temperature, strain and particle velocity have the forms given in (5.4), (5.5). Enforcing continuity of strain and particle velocity at the fan boundaries determines the constants A and B in (5.5) as well as the velocities $\dot{s}_1$ and $\dot{s}_2$ that define the boundaries of the fan. This calculation yields the following small-ε approximations for the dimensionless versions $\xi_1 = \dot{s}_1/a$ and $\xi_2 = \dot{s}_2/a$ of these velocities:

$$\xi_1 = 1 + \tfrac{1}{2} T_0 \gamma_T^2 \varepsilon^2 + O(\varepsilon^4), \qquad \xi_2 = 1 + \tfrac{1}{2} T_0 \gamma_T^2 \varepsilon^2 + \tfrac{1}{2} T_0 w_0 \gamma_T^3 \varepsilon^3 + O(\varepsilon^4). \tag{7.5}$$

Since necessarily $\xi_2 < \xi_1$, one must have $w_0 < 0$. Therefore, at least for small ε, the no-phase-change solution with a fan can only occur when the impact velocity is negative, corresponding to tensile loading. Thus in the absence of a phase transition, as in gas dynamics, compression induces a shock, while 'rarefaction' is described by continuous strain and temperature fields involving a fan. The ultimate strain and dimensionless temperature behind the fan are found to be given by

$$\bar{\gamma} = -\gamma_T w_0 + w_0 T_0 \gamma_T^3 \varepsilon^2/2 + O(\varepsilon^3), \qquad \bar{T} = T_0 + T_0 w_0 \gamma_T \varepsilon + O(\varepsilon^2). \tag{7.6}$$

for small ε. Comparison of $(7.4)_{2,3}$ and (7.6) shows that, to the first two orders in ε, the ultimate strain and temperature in the bar are related to the impact velocity and the initial temperature in the same way in the two types of solution that correspond

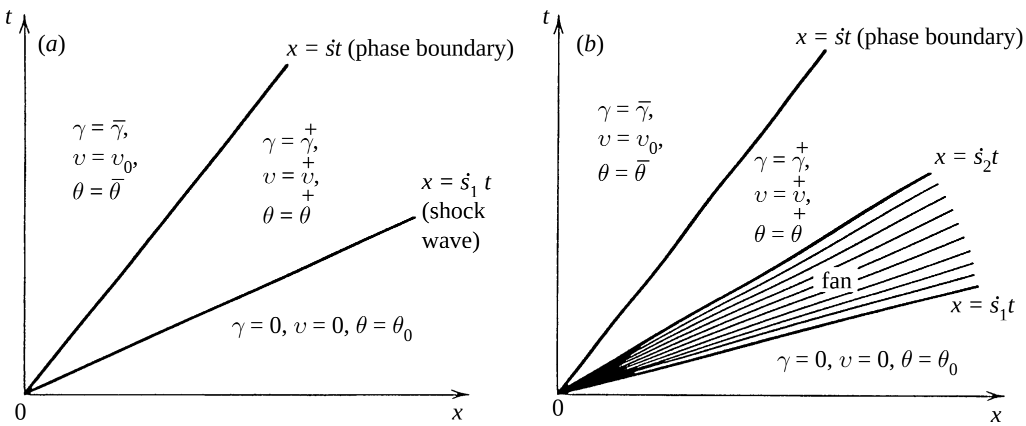

Figure 13. The x, t-plane for the impact problem in the case $\alpha \neq 0$ with a phase transition: (*a*) with shock wave; (*b*) with fan.

to the absence of a phase transition. A similar remark applies to the dimensionless velocities ξ_1 in $(7.4)_1$ and ξ_1, ξ_2 in (7.5).

(*b*) *Solutions with a phase-change*

When a phase transition does occur, there are again two types of solution: one with a phase boundary, a shock wave and no fan, another with a phase boundary, a fan and no shock wave; see figure 13.

We first discuss the former case, in which the solution involves a *phase boundary and a shock wave, but no fan*, as indicated in figure 13*a*. Enforcing the jump conditions at the phase boundary $x = \dot{s}t$ and at the shock wave $x = \dot{s}_1 t$ provides six equations for the determination of the six unknowns

$$\overset{\pm}{\gamma}, \quad \overset{+}{v}, \quad \overset{\pm}{\theta}, \quad \dot{s}_1$$

as functions of the phase boundary velocity $\dot{s}$, which in turn must ultimately be determined by the kinetic relation. While these equations cannot be solved explicitly in closed form, they readily provide approximations for the various fields when ε is small. To describe some of these approximations, it is helpful to rewrite $(3.8)_2$ in the form

$$\lambda_{\mathrm{T}} = \overset{\circ}{\lambda}_{\mathrm{T}} + a^2 T_{\mathrm{T}} \gamma_{\mathrm{T}}^3 \varepsilon, \tag{7.7}$$

where $\overset{\circ}{\lambda}_{\mathrm{T}}$ is the latent heat when $\alpha = 0$.

For small ε, the principal differences between the present phase-change solution (figure 13*a*) and the phase-change solution when $\alpha = 0$ (figure 4*b*) is that the entropy inequality at the shock wave is no longer trivial. To leading order in ε, the unknowns

$$\overset{\pm}{\gamma}, \quad \bar{\theta}, \quad \overset{+}{v}$$

of figure 13*a* are again given by (6.9)–(6.12), while

$$\overset{+}{\theta} = \theta_0 + O(\varepsilon).$$

When evaluated to the first non-trivial order in ε, the entropy inequality at the shock wave is found to require that

$$w_0 \geqslant \frac{\xi}{1-\xi^2}, \quad 0 \leqslant \xi < 1 \quad \text{(phase change with shock wave, no fan).} \tag{7.8}$$

Because of (7.8), one finds from (6.11) that the strain immediately behind the shock wave is compressive. When the dimensionless version of $\overset{+}{\theta}$ is calculated to the next order in ε, it is found to be given by

$$\overset{+}{T} = T_0 + \varepsilon\gamma_{\mathrm{T}} T_0 \left(w_0 - \frac{\xi}{1-\xi^2} \right) + O(\varepsilon^2), \quad 0 \leqslant \xi < 1, \tag{7.9}$$

so that, by (7.8), (7.9), the temperature is slightly increased by the passage of the shock.

We next turn to the case of a phase-change solution involving *a fan and a phase boundary, but no shock wave*, as indicated in figure 13*b*. Within the fan, the fields are of the form (5.4), (5.5). Enforcing the jump conditions at the phase boundary $x = \dot{s}t$ and continuity of the fields at the boundaries $x = \dot{s}_1 t$ and $x = \dot{s}_2 t$ of the fan provides nine equations for the determination of the nine unknowns

$$\overset{\pm}{\gamma}, \quad \overset{\pm}{\theta}, \quad \overset{+}{v}, \quad \dot{s}_1, \quad \dot{s}_2$$

and the constants A and B in (5.5) as functions of the phase boundary velocity $\dot{s}$. While these equations too cannot be solved explicitly in closed form, they also readily provide approximations for the various fields when ε is small.

For small ε, the principal difference between the present phase-change solution (figure 13*b*) and the phase-change solution when $\alpha = 0$ (figure 4*b*) is that the shock wave in the latter is now replaced by a narrow fan. To leading order in ε, the unknowns

$$\overset{\pm}{\gamma}, \quad \overline{\theta}, \quad \overset{+}{v}$$

in figure 13*b* are given by the formulas (6.9)–(6.12), while

$$\overline{\theta} = \theta_0 + O(\varepsilon).$$

The dimensionless velocities $\xi_1 = \dot{s}_1/a$ and $\xi_2 = \dot{s}_2/a$ corresponding to the boundaries of the fan are given in terms of the dimensionless phase boundary velocity $\xi = \dot{s}/a$ by

$$\xi_1 = 1 + \gamma_{\mathrm{T}}^2 T_0 \varepsilon^2/2 + O(\varepsilon^4), \quad \xi_2 = 1 + \gamma_{\mathrm{T}}^2 T_0 \varepsilon^2/2 + \gamma_{\mathrm{T}}^3 T_0 \left(w_0 - \frac{\xi}{1-\xi^2} \right) \varepsilon^3/2 + O(\varepsilon^4), \tag{7.10}$$

for small ε; recall that $0 \leqslant \xi < 1$. Since $\xi_2 < \xi_1$, (7.10) implies that the *reversed* version of (7.8) must hold in the present case:

$$w_0 \leqslant \frac{\xi}{1-\xi^2}, \quad 0 \leqslant \xi < 1 \quad \text{(phase change with fan, no shock wave).} \tag{7.11}$$

By (6.11), a surprising consequence of this inequality is that the strain $\overset{+}{\gamma}$ immediately behind the fan is *tensile*. Moreover, if one calculates the temperature $\overset{+}{\theta}$ to the next order in ε, one again obtains (7.9), which, together with (7.11), implies that the temperature is now *decreased* slightly by the passage of the fan. These two results are qualitatively opposite to their counterparts in the phase-change solution with a shock wave but no fan.

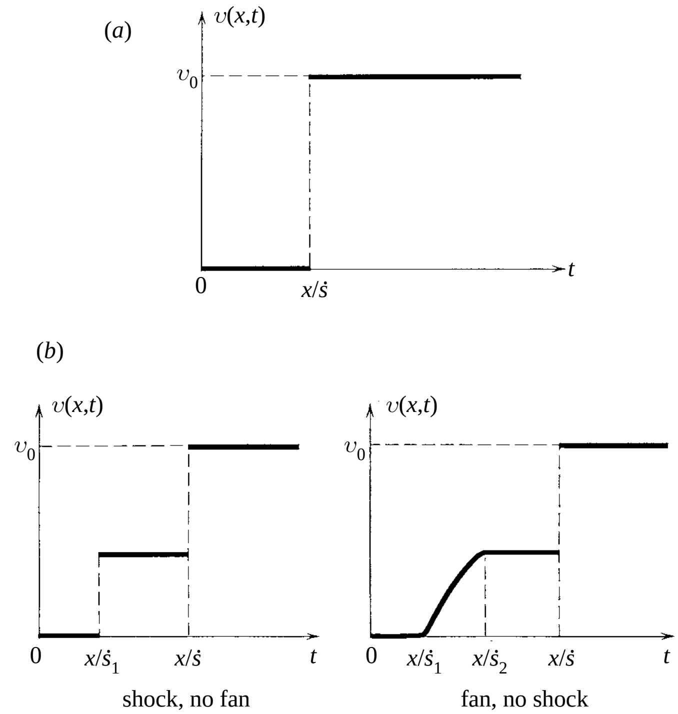

Figure 14. Schematic plot of velocity time-history of a particle in the case $\alpha \neq 0$: (*a*) without phase change; (*b*) with phase change.

For phase-change solutions of both types, one must enforce restrictions arising from strain-range requirements and the entropy inequality at the phase boundary. In the special case

$$\mathring{\lambda}_{\mathrm{T}} = 0$$

and to leading order in ε, these requirements restrict the totality of solutions to the same admissible region R shown in figure 5 for the case $\alpha = 0$, $\lambda_{\mathrm{T}} = 0$. If one draws in this region the curve Γ defined by $w_0 = \xi/(1-\xi^2)$, one finds that this curve lies above the curve $f = 0$ forming the lower boundary of R and divides R into two parts. By (7.8), points in R *above* Γ correspond to phase-change solutions with a shock wave and no fan, while by (7.11), points in R below Γ yield phase-change solutions with a fan but no shock wave. Which of these alternatives actually occurs depends on where, within the shaded region of figure 5 and relative to the curve Γ, the curve K (figure 6) generated by the kinetic relation lies. Since the phase change accompanied by a fan leads to the counter-intuitive results of tensile strain and a drop in temperature behind the fan, one might expect the corresponding solution *not* to occur, though there is no fundamental reason why this should be so.

Finally, for either type of phase-change solution, one must calculate the driving traction at the phase boundary and impose the kinetic relation to determine the phase boundary velocity $\dot{s}$ in terms of the impact velocity v_0 and the initial temperature θ_0. We omit this calculation, since the numerical procedure involved is straightforward.

Figures 14*a*, *b* are schematic plots of particle velocity $v(x,t)$ at fixed x as a function of time for the respective cases in which a phase change is absent (figure 14*a*) or present (figures 14*b*). The impact velocity is assumed to be positive in each case, corresponding to compression. The solution without a phase change therefore does not involve a fan; there may or may not be a fan when the phase transition occurs. Figure 14 is the counterpart for ε small and positive of figure 8, which applies when $\varepsilon = 0$.

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